

## Section 6.4: General Logarithmic and Exponential Functions

### General Exponential Functions

If  $a > 0$ , and  $r \in \mathbb{Q}$  (rational number), Then

$$a^r = (e^{\ln a})^r = e^{r \cdot \ln a}. \text{ So, we define}$$

$$a^x = e^{x \cdot \ln a} \quad \text{for all real numbers } x.$$

- We call  $a^x$  an exponential function with base  $a$ .
- Since  $e^x > 0$  for all  $x$ . Then  $a^x > 0$  for all  $x$ , as well.
- $\ln(a^r) = r \cdot \ln(a)$  for all real numbers  $r$ .

Laws of exponents: Let  $x, y \in \mathbb{R}$ ,  $a, b > 0$

$$\textcircled{1} \quad a^{x+y} = a^x a^y$$

$$\textcircled{2} \quad a^{x-y} = \frac{a^x}{a^y}$$

$$\textcircled{3} \quad (a^x)^y = a^{xy}$$

$$\textcircled{4} \quad (ab)^x = a^x b^x$$

$$\text{Proof } \textcircled{1} \quad a^{x+y} = e^{(x+y)\ln(a)} = e^{x\ln a + y\ln a} = e^{x\ln a} e^{y\ln a} = a^x \cdot a^y$$

$$\textcircled{3} \quad (a^x)^y = e^{y \cdot \ln(a^x)} = e^{y \cdot x \cdot \ln a} = e^{xy \cdot \ln(a)} = a^{xy}$$

$$\text{Differentiation: } \frac{d}{dx} a^x = \frac{d}{dx} (e^{x \ln a}) = e^{x \ln a} \cdot \frac{d}{dx} (x \ln a) = a^x \cdot \ln a$$

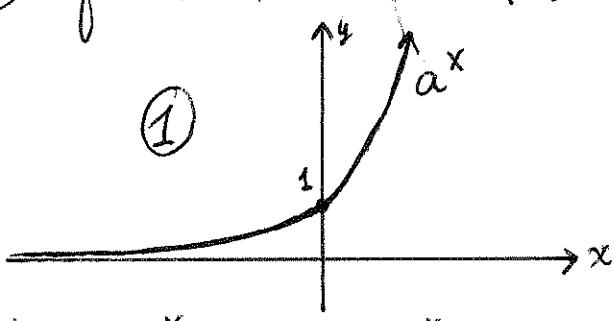
So, 
$$\boxed{\frac{d}{dx} a^x = a^x \ln a}$$

Examples: ①  $\frac{d}{dx} \left(\frac{3}{2}\right)^x = \left(\frac{3}{2}\right)^x \cdot \ln\left(\frac{3}{2}\right)$

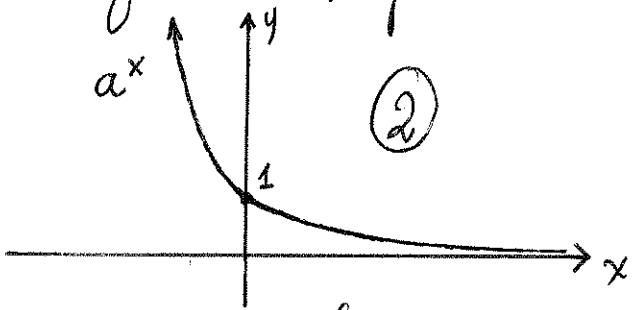
②  $\frac{d}{dx} 5^{x^2-1} = 5^{x^2-1} \cdot \ln(5) \cdot \frac{d}{dx}(x^2-1) = 5^{x^2-1} \ln(5) \cdot 2x$

Exponential Graphs. let  $f(x) = a^x = e^{x \cdot \ln a}$ ,  $f'(x) = a^x \cdot \ln a$

- ① if  $a > 1$ , Then  $\ln a > 0$ , and  $f'(x) > 0$  for all  $x \Rightarrow f$  is increasing
- ② if  $0 < a < 1$ , Then  $\ln a < 0$ , and  $f'(x) < 0$  for all  $x \Rightarrow f$  is decreasing



$$\lim_{x \rightarrow -\infty} a^x = 0, \lim_{x \rightarrow +\infty} a^x = +\infty$$



$$\lim_{x \rightarrow -\infty} a^x = +\infty, \lim_{x \rightarrow +\infty} a^x = 0$$

Exponential Integrals  $\boxed{\int a^x dx = \frac{a^x}{\ln a} + C, \quad a \neq 1}$

Examples ①  $\int 5^x dx = \frac{5^x}{\ln 5} + C$

②  $\int e^x dx = \frac{e^x}{\ln e} = e^x + C \quad \checkmark$

③  $\int 6^{\sin x} \cos x dx \quad \text{Let } u = \sin x, du = \cos x dx$

$$\int 6^u du = \frac{6^u}{\ln 6} = \frac{6^{\sin x}}{\ln 6} + C$$

## Power rule Vs. Exponential rule

- A proof of the power rule:  $\frac{d}{dx} x^n = n x^{n-1}$

Let  $y = x^n$ ; Then  $\ln|y| = \ln|x^n| = n \ln|x|$ .

$$\text{Thus, } \frac{d}{dx} \ln|y| = \frac{d}{dx} (n \ln|x|) \Rightarrow \frac{y'}{y} = \frac{n}{x}$$

$$\Rightarrow y' = \frac{n y}{x} = \frac{n \cdot x^n}{x} = n x^{n-1}. \text{ Thus } \frac{d}{dx} x^n = n x^{n-1}$$

- Using Log. differentiation on functions of the form  $(f(x))^{g(x)}$

Example: Differentiate  $y = x^{\sqrt{x}}$

Caution!  $x^{\sqrt{x}}$  is NOT a power function ( $x^n$ ) Since  
The Power is NOT constant, and it is NOT an  
Exponential function ( $a^x$ ) since the base is NOT constant.

Let  $y = x^{\sqrt{x}}$ ,  $\ln y = \ln(x^{\sqrt{x}}) = \sqrt{x} \ln(x)$ . Then

$$\frac{d}{dx} \ln y = \frac{d}{dx} (\sqrt{x} \ln x) \Rightarrow \frac{y'}{y} = \frac{1}{2\sqrt{x}} \ln x + \sqrt{x} \cdot \frac{1}{x}$$

$$\text{Thus, } y' = y \left( \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \right)$$

$$= x^{\sqrt{x}} \left( \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \right).$$

## General Logarithmic Functions

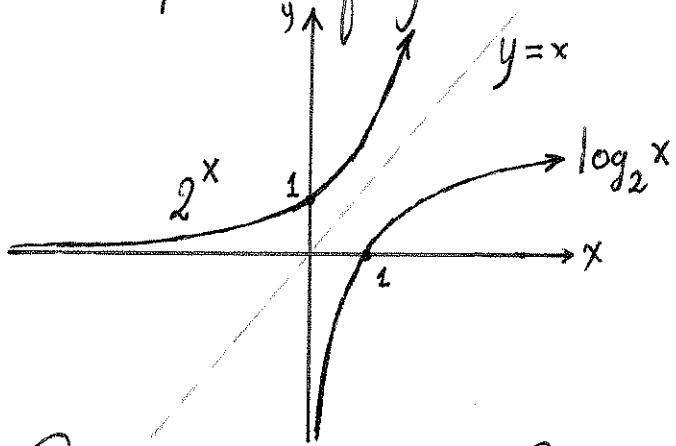
We have just seen that for  $a > 0$ , ( $a \neq 1$ ),  $f(x) = a^x$  is a one-to-one function, and thus it must have an inverse.

The inverse of  $f(x) = a^x$  is  $f^{-1}(x) = \log_a x$ . So,

$$\log_a x = y \iff a^y = x$$

Cancellation Equations:  $a^{\log_a x} = x$ , and  $\log_a a^x = x$

Example: if  $f(x) = 2^x$ , Then  $f^{-1}(x) = \log_2 x$ . Graphically,



$2^x$  is very rapidly increasing

$\Rightarrow \log_2 x$  is very slowly increasing

Change of Base formula: for  $a > 0$ ,  $a \neq 1$ ,

$$\log_a x = \frac{\ln x}{\ln a}$$

Proof. Let  $y = \log_a x$ ; Then  $a^y = a^{\log_a x} = x$

$$a^y = x \Rightarrow \ln(a^y) = \ln(x) \Rightarrow y \cdot \ln(a) = \ln(x)$$

$$\Rightarrow y = \frac{\ln x}{\ln a}; \text{ So, } \log_a x = \frac{\ln x}{\ln a}$$

Differentiation:  $\frac{d}{dx} \log_a x = \frac{1}{x \cdot \ln a}$

$$\text{Examples } \textcircled{1} \frac{d}{dx} \log_6 (2x - \sec x) = \frac{1}{(2x - \sec x) \cdot \ln 6} \cdot \frac{d}{dx} (2x - \sec x)$$

$$= \frac{2 - \sec x \tan x}{(2x - \sec x) \cdot \ln 6}$$

$$\textcircled{2} \frac{d}{dx} 2^x \cdot \log_2 x = 2^x \cdot \ln 2 \cdot \log_2 x + 2^x \cdot \frac{1}{x \cdot \ln 2}$$

Remark:  $\log_e x = \frac{\ln x}{\ln e} = \ln x$ . So  $\log_e = \ln$ .

The number e as a limit:

Let  $f(x) = \ln x$ . Then  $f'(x) = \frac{1}{x}$ , and  $f'(1) = 1$ ; also

$$f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{\ln(1+h) - \ln(1)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \cdot \ln(1+h)$$

$$\text{So, } f'(1) = \lim_{h \rightarrow 0} \ln[(1+h)^{\frac{1}{h}}] = 1 \quad (\text{Raise e to both sides})$$

$$\lim_{h \rightarrow 0} e^{\ln[(1+h)^{\frac{1}{h}}]} = e^1 \Rightarrow \lim_{h \rightarrow 0} (1+h)^{\frac{1}{h}} = e$$

Therefore,  $e = \boxed{\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}}}$ ;

If we let  $n = \frac{1}{x}$ , Then  $e = \boxed{\lim_{n \rightarrow \infty} (1+\frac{1}{n})^n}$ .